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GREEN'S FUNCTION AND ENERGY INTEGRALS FOR MAGNETO-ELASTIC SURFACE WAVES IN PERFECTLY CONDUCTING MEDIA

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Abstract—Propagation of magneto-elastic Rayleigh waves from a unit line force in a perfectly conducting elastic half-space in a magnetic field is considered in this paper. Green's function for magneto-elastic Rayleigh waves is constructed by using the theory of spectral operators. A unified self-adjoint operator for both the elastic and electromagnetic disturbances is introduced. The self-adjointedness of the operator is then invoked to construct a solution of the inhomogeneous problem through eigenfunction of a corresponding homogeneous problem. Energy integrals for magneto-elastic surface waves and electro-magnetic disturbances similar to the more familiar elastic energy integrals are derived and relationships between the energy integrals are discussed in some detail.

1. INTRODUCTION

The interaction between electromagnetic fields and deformable solids has received considerable attention in recent years for possible applications in high energy devices such as magnetically levitated vehicles (Maglev), MHD generators, fusion reactors, magnetic launcher, superconducting magnetic energy storage (SMES), and magnetic forming devices (Moon, 1984). Especially, the propagation of mechanical or thermo-mechanical waves through a magnetic field has been a topic for many investigators into the detection of flaws in ferrous and nonferrous metals, optical acoustics and geophysics. Knopoff (1955) studied the effect of the earth's magnetic field on the propagation of seismic waves in the conducting core of the earth. Dunkin and Eringen (1963) studied the problem of plane waves traveling through an infinite medium and an infinite plate in the presence of large magnetostatic and electrostatic fields. Other works on the topic include Paria (1962), Wilson (1963), Chian and Moon (1981), Verma (1986), and Lee *et al.* (1990). More recently, Massalas and Tsolakidis (1990) studied plane magneto-thermo-elastic wave propagation in a prestressed body.

Most of the aforementioned studies, however, are concerned with the propagation of body waves. Propagation of surface waves in elastic conductors has not received much attention despite some potential applications in nondestructive material characterization of advanced electromagnetic materials. The only exception is probably a series of works by Kaliski and Rogula (1960, 1961). Recently the propagation of magneto-elastic Rayleigh waves in a perfectly conducting elastic half-space similar to the problem treated by Kaliski and Rogula (1960) has been investigated by Lee and Its (1991, 1992) to study the influence of the magnetic field and magnetic properties on the parameters of Rayleigh waves.

Any meaningful application of surface waves to a nondestructive measurement of the electromagnetic and/or mechanical parameters requires investigations in the framework of dynamics of inhomogeneous media. It is known, however, that the solution of many inhomogeneous problems such as the problems of surface wave scattering at weak inhomogeneities (Snieder, 1987) or the problems of reflection and transmission at strong inhomogeneities (Its and Yanovskaya, 1985; Its, 1991) are based on the construction of Green's functions for homogeneous media. Unfortunately, no Green's functions for coupled magneto-elastic surface waves have been reported in the literature. Most studies on the

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coupled magneto-elastic wave propagation have invariably been concerned with homogeneous (i.e. source-free) problems. Therefore, discussions have been confined to the dispersion relations, and thus limiting the applicability of the solution to practical situations.

In order to provide a framework for further studies on the reflection and transmission of magneto-elastic waves in magneto-elastic media with various material inhomogeneities, the Green's function for surface waves propagating in a perfectly conducting half-space is constructed in this paper for the case when the half-space containing a unit mechanical singularity is permeated by a transverse magnetic field. A unified self-adjoint operator for both the elastic and electromagnetic disturbances is introduced. The self-adjointedness of the operator is then invoked to construct a solution of the inhomogeneous problem through eigenfunctions of the corresponding homogeneous problem. By analogy with the energy integrals known for purely mechanical surface waves, magneto-elastic energy integrals for coupled elastic and electro-magnetic disturbances are also derived and relationships between energy integrals are discussed in some detail.

2. FUNDAMENTAL EQUATIONS AND STATEMENT OF THE PROBLEM

Consider an electrically conducting elastic half-space ($-\infty < x < \infty$, $-\infty < y < \infty$, $0 < z < \infty$) in contact with a vacuum which is permeated by a magnetic field. Equations governing the propagation of small elastic disturbances are

$$\nabla \cdot \mathbf{T} + \rho(\mathbf{f} - \ddot{\mathbf{u}}) + \mathbf{J} \times \mathbf{B} = 0, \tag{1}$$

where $\mathbf{T}$ is the stress tensor, $\mathbf{u}$ the mechanical displacement, ρ the mass density, and $\mathbf{f}$ is the body force of non-magnetic origin. The last term in the above equation is the Lorentz force due to the electromagnetic field. The electromagnetic field is governed by Maxwell's equations

$$\nabla \times \mathbf{E} = -\dot{\mathbf{B}}, \quad \nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{H} = \mathbf{J}, \tag{2}$$

where $\mathbf{H}$ is the magnetic field, $\mathbf{B}$ the magnetic induction, $\mathbf{E}$ the electric field, and $\mathbf{J}$ is the current density. In the above equations, we have neglected the displacement current. Equations (1) and (2) are supplemented by the following constitutive equations:

$$\mathbf{B} = \mu\mathbf{H}, \quad \mathbf{J} = \sigma(\mathbf{E} + \dot{\mathbf{u}} \times \mathbf{B}), \tag{3}$$

$$\mathbf{T} = \lambda(\mathrm{tr}\,\mathbf{e})\mathbf{I} + 2G\mathbf{e}, \quad \mathbf{e} = (\nabla\mathbf{u} + \mathbf{u}\nabla)/2, \tag{4}$$

where μ is the magnetic permeability, σ the electric conductivity, $\mathbf{e}$ the elastic strain tensor, λ and G are the Lamé constants, and $\mathbf{I}$ is the identity tensor. Superscript dots, $\dot{}$ and $\ddot{}$, denote the first and second differentiations with respect to time, respectively. $\mathbf{u}\nabla$ denotes a transposition of the displacement gradient $\nabla\mathbf{u}$. The boundary conditions are

$$\mathbf{n} \times [\![\mathbf{E} + \dot{\mathbf{u}} \times \mathbf{B}]\!] = 0, \quad \mathbf{n} \cdot [\![\mathbf{B}]\!] = 0, \quad \mathbf{n} \times [\![\mathbf{H}]\!] = 0, \tag{5}$$

$$\mathbf{n} \cdot [\![\mathbf{T} + \mathbf{T}^E]\!] = 0, \tag{6}$$

where $\mathbf{n}$ is the unit outward normal vector, $[\![\ \ ]\!]$ denotes the jump between the values from the positive and negative sides of the discontinuity surface, and Maxwell's stress tensor $\mathbf{T}^E$ in (6) is given by

$$\mathbf{T}^E = \mu(\mathbf{H}\mathbf{H} - \tfrac{1}{2}H^2\mathbf{I}), \tag{7}$$

where $H = \mathbf{H} \cdot \mathbf{H}$.

Another set of governing equations is needed for the vacuum and may be obtained from eqns (2) by setting $\mu = \mu_0$ and $\sigma = 0$ in eqns (3). All field variables in the vacuum will be denoted by a bar on the corresponding variables in the magneto-elastic medium. Linearization is carried out on a small perturbation of a primary bias field $\mathbf{H}^0$ due to the electro-magnetic-mechanical interaction, e.g.

$$\mathbf{H}(\mathbf{x},t) = \mathbf{H}^0+\mathbf{h}(\mathbf{x},t), \quad \mathbf{E}(\mathbf{x},t) = 0+\mathbf{e}(\mathbf{x},t), \quad \mathbf{J}(\mathbf{x},t) = 0+\mathbf{j}(\mathbf{x},t), \tag{8}$$

where the field quantities in lowercase letters are assumed to be small such that their products can be neglected.

Consider a unit line force located in the perfectly conducting half-space, $\mathbf{f} = \delta_n\delta(x-x^0)\delta(z-z^0)\delta(t-\tau)$, and restrict ourselves to a case when the half-space is permeated by a normal magnetic field $\mathbf{H} = (0, 0, H)$. One can present all fields under consideration and the unit force in Fourier transformation form such that

$$\mathbf{w}(\mathbf{x},\mathbf{x}^0,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega(t-\tau)}\mathbf{v}(\mathbf{x},\mathbf{x}^0,\omega)\,\mathrm{d}\omega, \tag{9}$$

$$\delta(t-\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega(t-\tau)}\,\mathrm{d}\omega. \tag{10}$$

Equations of motion (1) can then be written by using the linearized Maxwell's equations in the following form (Lee and Its, 1991):

$$\mathbf{L}^+\mathbf{U}^n - \frac{\delta_n\delta(\mathbf{x}-\mathbf{x}^0)}{\rho} = \omega^2\mathbf{U}^n, \tag{11}$$

where $\mathbf{U}^n = [u_1^n, u_3^n]^T$ and the components of the operator $\mathbf{L}^+$ are given as follows:

$$L_{11}^+ = -\frac{1}{\rho}\left[(\lambda+2G)\frac{\partial^2}{\partial x^2}+G\frac{\partial^2}{\partial z^2}+\mu H^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)\right],$$

$$L_{21}^+ = L_{12}^+ = -\frac{1}{\rho}(\lambda+G)\frac{\partial}{\partial x}\frac{\partial}{\partial z},$$

$$L_{22}^+ = -\frac{1}{\rho}\left[G\frac{\partial^2}{\partial x^2}+(\lambda+2G)\frac{\partial^2}{\partial z^2}\right]. \tag{12}$$

The perturbation of the electro-magnetic field in the half-space caused by elastic waves from the source is related to elastic displacement as follows (Lee and Its, 1991):

$$h_1^n = H\frac{\partial u_1^n}{\partial z}, \quad h_3^n = -H\frac{\partial u_1^n}{\partial x}, \quad e_2^n = i\omega\mu H u_1^n. \tag{13}$$

In the vacuum ($0 \leqslant z < -\infty$), the perturbation of the electro-magnetic field propagating from the half-space satisfies the following equation:

$$L^-\bar{h}_3^n = \omega^2\bar{h}_3^n, \tag{14}$$

where

$$L^- = -\frac{1}{\varepsilon_0\mu_0}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right). \tag{15}$$

Other components of the electro-magnetic field are connected with $\bar{h}_3^n$ by the relations

$$\frac{\partial \bar{e}_2^n}{\partial x} = -i\omega\mu_0\bar{h}_3^n, \quad \frac{\partial \bar{h}_3^n}{\partial z} = -\frac{\partial \bar{h}_1^n}{\partial x}. \tag{16}$$

Boundary conditions (5) and (6) at $z = 0$ can be rewritten as follows:

$$G\left(\frac{\partial u_1^n}{\partial z} + \frac{\partial u_3^n}{\partial x}\right) = 0,$$

$$(\lambda+2G)\frac{\partial u_3^n}{\partial z} + \lambda\frac{\partial u_1^n}{\partial x} = -\mu H(h_3^n - \bar{h}_3^n),$$

$$h_1^n = \bar{h}_1^n, \quad e_2^n = \bar{e}_2^n, \quad \mu h_3^n = \mu_0\bar{h}_3^n. \tag{17}$$

We now wish to find a solution to eqns (11) and (14) which are subjected to the coupled boundary conditions (17).

3. GREEN'S FUNCTION FOR MAGNETO-ELASTIC RAYLEIGH WAVES

By introducing a vector $\mathbf{G}^n = [u_1^n, u_3^n, \bar{h}_3^n]^T$, eqns (11) and (14) can be combined and presented as

$$\tilde{\mathbf{L}}\mathbf{G}^n - \omega^2\mathbf{G}^n = \frac{\delta_n\delta(x-x^0)\delta(z-z^0)}{\rho}, \tag{18}$$

where the components of the operator

$$\tilde{\mathbf{L}} = \begin{pmatrix} L_{11}^+ & L_{12}^+ & 0 \\ L_{21}^+ & L_{22}^+ & 0 \\ 0 & 0 & L^- \end{pmatrix} \tag{19}$$

are defined by eqns (12) and (15). Due to the assumption of perfect conductivity, the off-diagonal terms connecting the two half-spaces in eqn (19) are rendered zero. The solutions of (18) in two half-spaces, however, are coupled through the boundary conditions (17).

We now consider the Fourier transformation of $\mathbf{G}^n$ and $\delta(x-x^0)$ such that

$$\mathbf{G}^n = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathbf{V}^n(z, z^0, \xi)e^{-i\xi(x-x^0)}\,\mathrm{d}\xi, \tag{20}$$

$$\delta(x-x^0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\xi(x-x^0)}\,\mathrm{d}\xi. \tag{21}$$

As will be shown later $\mathbf{V}^n(z, z^0, \xi)$ has singularities on the real axis ξ; henceforth we will consider (20) as a contour integral at the plane of complex ξ. The contour of integration is taken along the real axis ξ and around the poles along a semi-circle in the upper half-plane. Inserting (20) and (21) into (18) and (17) results in a one-dimensional inhomogeneous boundary value problem which can be described by the following equation:

$$\mathbf{L}\mathbf{V}^n - \omega^2\mathbf{V}^n = \delta_n\frac{\delta(z-z^0)}{\rho}, \tag{22}$$

where

$$L_{11} = -\frac{1}{\rho}\left[-(\lambda+2G)\xi^2+G\frac{d^2}{dz^2}+\mu H^2\left(\frac{d^2}{dz^2}-\xi^2\right)\right],$$

$$L_{12} = L_{21} = -\frac{1}{\rho}\left[-i\xi(\lambda+G)\frac{d}{dz}\right],$$

$$L_{22} = -\frac{1}{\rho}\left[(\lambda+2G)\frac{d^2}{dz^2}-G\xi^2\right],$$

$$L_{33} = -\frac{1}{\varepsilon_0\mu_0}\left[\frac{d^2}{dz^2}-\xi^2\right],$$

$$L_{13} = L_{23} = L_{31} = L_{32} = 0, \tag{23}$$

and by the following boundary conditions at $z = 0$:

$$G\left(\frac{dV_1^n}{dz}-i\xi V_2^n\right) = 0, \tag{24}$$

$$(\lambda+2G)\frac{dV_2^n}{dz}-i\xi\lambda V_1^n+i\xi\mu H^2 V_1^n = \mu H V_3^n, \tag{25}$$

$$\mu_0 V_3^n = i\xi\mu V_1^n H. \tag{26}$$

Taking into account that V_3 is defined for $z < 0$ while V_1 and V_2 are defined for $z > 0$, one can introduce a scalar product in Hilbert space $\mathbb{H}\in[V_1, V_2, V_3]^T$ as

$$\langle\langle \mathbf{V}, \tilde{\mathbf{V}}\rangle\rangle = k\int_{-\infty}^{0}\varepsilon_0\mu_0 V_3\tilde{V}_3^*\,dz+\int_0^{\infty}\rho(V_1\tilde{V}_1^*+V_2\tilde{V}_2^*)\,dz. \tag{27}$$

With respect to the scalar product (27) one can show (see Appendix) that the operator constructed from the left-hand side of eqn (22) and boundary conditions (24)–(26) is self-adjoint if $k = \mu_0/\xi^2$, therefore, the solution of (22)–(26) can be represented by the superposition of the eigenfunctions from the corresponding homogeneous boundary value problem (Titchmarsh, 1962) to our original problem (22)–(26) as the following:

$$V_m^n = \sum_{k=1}^{K(\xi)} c_k^n\tilde{V}_{km}^n+\int_{-\infty}^{\infty} c^n(\kappa)\tilde{V}_m^n(\kappa, z)\,d\kappa, \tag{28}$$

where the functions $\tilde{V}_{km}^n$ correspond to a discrete spectrum of eigenvalues ω_k, and functions $\tilde{V}_m^n(\kappa, z)$ correspond to a continuous spectrum of eigenvalues κ.

In Lee and Its (1992) it has been shown that the homogeneous boundary problem under consideration has one discrete solution in a wide interval of variation of μ and H. Here we consider a perfectly conducting material with a magnetic permeability for which a discrete solution does exist in a given original magnetic field. Then the solution describes a magneto-elastic surface wave, similar to the purely elastic Rayleigh wave in an elastic medium, and an electro-magnetic disturbance excited by the surface wave.

Let $\tilde{\mathbf{V}}(z)$ be a solution (corresponding to eigenvalue $\tilde{\omega}^2$) of the homogeneous equation

$$\mathbf{L}\tilde{\mathbf{V}}(z)-\tilde{\omega}^2\tilde{\mathbf{V}}(z) = 0 \tag{29}$$

and boundary conditions (24)–(26). Then the solution of (22) can be written as

$$\mathbf{V}^n = c_n\tilde{\mathbf{V}}(z) + \int_{-\infty}^{\infty} c_n(\kappa)\tilde{\mathbf{V}}(\kappa, z)\,\mathrm{d}\kappa, \tag{30}$$

where $\tilde{\mathbf{V}}(\kappa, z)$ are normalized eigenfunctions of continuous spectrum. The right-hand side of eqn (22) can also be expressed as

$$\delta_n \frac{\delta(z-z^0)}{\rho} = d_n\tilde{\mathbf{V}}(z) + \int_{-\infty}^{\infty} d_n(\kappa)\tilde{\mathbf{V}}(\kappa, z)\,\mathrm{d}\kappa. \tag{31}$$

Multiplying (31) with $\tilde{\mathbf{V}}(z)$ and using the orthogonality of $\tilde{\mathbf{V}}(z)$ and $\tilde{\mathbf{V}}(\kappa, z)$ leads to the relation

$$\left\langle\!\!\left\langle \frac{\delta_n\delta(z-z^0)}{\rho}, \tilde{\mathbf{V}}(z) \right\rangle\!\!\right\rangle = d_n\|\tilde{\mathbf{V}}\|^2. \tag{32}$$

Since the source is located in the conducting half-space ($z \geq 0$), we obtain another relation as

$$\left\langle\!\!\left\langle \delta_n \frac{\delta(z-z^0)}{\rho}, \tilde{\mathbf{V}}(z) \right\rangle\!\!\right\rangle = \int_0^{\infty} \delta_n\delta(z-z^0)\tilde{\mathbf{V}}^*(z)\,\mathrm{d}z = \tilde{V}^*_{+n}(z^0), \tag{33}$$

where

$$\tilde{\mathbf{V}}^*_{+}(z^0) = [\tilde{V}^*_1(z^0), \tilde{V}^*_2(z^0), 0]^{\mathrm{T}}. \tag{34}$$

Hence the coefficient d_n can be determined by

$$d_n = \frac{\tilde{V}^*_{+n}(z^0)}{\|\tilde{\mathbf{V}}\|^2}.$$

Inserting (30) into (22) leads to the relation

$$\mathbf{L}c_n\tilde{\mathbf{V}}(z) - \omega^2 c_n\tilde{\mathbf{V}}(z) = d_n\tilde{\mathbf{V}}(z). \tag{35}$$

Taking into account (29) we can rewrite eqn (35) as

$$c_n(\tilde{\omega}^2 - \omega^2)\tilde{\mathbf{V}}(z) = d_n\tilde{\mathbf{V}}(z)$$

or

$$c_n = \frac{\tilde{V}^*_{+n}(z^0)}{\|\tilde{\mathbf{V}}\|^2(\tilde{\omega}^2 - \omega^2)}. \tag{36}$$

Similarly we obtain

$$c_n(\kappa, z^0) = \frac{\tilde{V}^*_{+n}(\kappa, z^0)}{\kappa^2(\xi) - \omega^2}. \tag{37}$$

Finally, if we insert eqns (36), (37) into (30), substitute the resulting equation into (20), and assume the double integral decreases as $x \to \infty$, we can present the discrete part of the solution to eqn (18) by

$$\mathbf{G}^n = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\tilde{V}^*_{+n}(z^0)\tilde{\mathbf{V}}(z)e^{-i\xi(x-x^0)}\,\mathrm{d}\xi}{\|\tilde{\mathbf{V}}\|^2(\tilde{\omega}^2 - \omega^2)}. \tag{38}$$

Eigenfunction $\tilde{\mathbf{V}}(z)$ in the medium ($z \geqslant 0$) has the form (Lee and Its, 1991):

$$\tilde{\mathbf{V}}(z) = \mathbf{A}e^{-\alpha z}+\mathbf{B}e^{-\beta z}, \tag{39}$$

where α and β are positive roots of the characteristic equation corresponding to (29). Therefore, a pole

$$\tilde{\omega}^2(\xi) = \omega^2 \tag{40}$$

is the only singular point of the expression under integral (38) which can be evaluated using the residue as

$$\mathbf{G}^n = i\tilde{V}^*_{+n}(z^0)\tilde{\mathbf{V}}(z)\frac{e^{-i\xi(x-x^0)}}{\|\tilde{\mathbf{V}}\|^2 2\omega \left.\dfrac{\partial\tilde{\omega}}{\partial\xi}\right|_{\xi=\xi}}. \tag{41}$$

Then the general solution of (1)–(7) in a perfectly conducting elastic half-space in a transverse magnetic field subjected to a unit line force has the form

$$\mathbf{w}(\mathbf{x},\mathbf{x}^0,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega(t-t)}\mathbf{G}^n(\mathbf{x},\mathbf{x}_0,\omega)\,\mathrm{d}\omega, \tag{42}$$

where the vector $\mathbf{w}$ consists of two components of displacement in the medium and one component of magnetic perturbation in the vacuum.

4. INTEGRAL REPRESENTATION

Let us take into account a shift between the vertical and horizontal components of displacement and redefine $\tilde{\mathbf{V}}$ as

$$\tilde{\mathbf{V}} = [-iv_1, v_2, v_3]^{\mathrm{T}}. \tag{43}$$

Equation (29) can then be rewritten in terms of the functions v_1, v_2 and v_3 in the form

$$-\frac{1}{\rho}\left[-(\lambda+2G)\xi^2 v_1+G\frac{\mathrm{d}^2v_1}{\mathrm{d}z^2}+\mu H^2\left(\frac{\mathrm{d}^2v_1}{\mathrm{d}z^2}-\xi^2v_1\right)-\xi(\lambda+G)\frac{\mathrm{d}v_2}{\mathrm{d}z}\right] = \tilde{\omega}^2v_1 \tag{44}$$

$$-\frac{1}{\rho}\left[(\lambda+2G)\frac{\mathrm{d}^2v_2}{\mathrm{d}z^2}-G\xi^2v_2-\xi(\lambda+G)\frac{\mathrm{d}v_1}{\mathrm{d}z}\right] = \tilde{\omega}^2v_2 \tag{45}$$

$$-\frac{1}{\varepsilon_0\mu_0}\left[\frac{\mathrm{d}^2v_3}{\mathrm{d}z^2}-\xi^2v_3\right] = \tilde{\omega}^2v_3 \tag{46}$$

and the boundary conditions (24)–(26) take the form

$$G\left(\frac{\mathrm{d}v_1}{\mathrm{d}z}+\xi v_2\right) = 0, \tag{47}$$

$$(\lambda+2G)\frac{\mathrm{d}v_2}{\mathrm{d}z}-\xi\lambda v_1+\xi\mu H^2v_1 = \mu Hv_3, \tag{48}$$

$$\mu_0 v_3 = \xi\mu v_1 H. \tag{49}$$

Multiplying eqns (44), (45) and (46) by ρv_1^*, ρv_2^*, and $(\varepsilon_0\mu_0^2/\xi^2)v_3^*$ respectively, integrating the resulting equations with respect to z from $-\infty$ to ∞, and taking into account the self-adjointedness of L leads to the following relations:

$$\bar{\omega}^2 I_1 - \xi^2 I_2 - \xi I_3 - I_4 + \frac{\bar{\omega}^2}{\xi^2} I_5 - \xi^2 I_6 - I_7 - I_8 - \frac{1}{\xi^2} I_9 = 0, \tag{50}$$

where $I_1 - I_4$ are the so-called energy integrals of the Rayleigh wave in elastic media (Aki and Richards, 1980) and are given by

$$I_1 = \int_0^\infty \rho[v_1^2 + v_2^2]\, \mathrm{d}z,$$

$$I_2 = \int_0^\infty \mu v_2^2\, \mathrm{d}z + \int_0^\infty (\lambda + 2G) v_1^2\, \mathrm{d}z,$$

$$I_3 = \int_0^\infty \mu\left(\frac{\mathrm{d}v_1}{\mathrm{d}z} v_2 + 2\frac{\mathrm{d}v_2}{\mathrm{d}z} v_1\right) \mathrm{d}z - \int_0^\infty (\lambda + 2G)\frac{\mathrm{d}v_2}{\mathrm{d}z} v_1\, \mathrm{d}z,$$

$$I_4 = \int_0^\infty \mu\left(\frac{\mathrm{d}v_1}{\mathrm{d}z}\right)^2 \mathrm{d}z + \int_0^\infty (\lambda + 2G)\left(\frac{\mathrm{d}v_2}{\mathrm{d}z}\right)^2 \mathrm{d}z. \tag{51}$$

Integrals $I_5 - I_9$ have the form

$$I_5 = \int_{-\infty}^0 \varepsilon_0 \mu_0^2 v_3^2\, \mathrm{d}z,$$

$$I_6 = \int_{-\infty}^0 \mu H^2 v_1^2\, \mathrm{d}z,$$

$$I_7 = \int_{-\infty}^0 \mu_0 v_3^2\, \mathrm{d}z,$$

$$I_8 = \int_{-\infty}^0 \mu H^2 \left(\frac{\mathrm{d}v_1}{\mathrm{d}z}\right)^2 \mathrm{d}z,$$

$$I_9 = \int_{-\infty}^0 \mu_0 \left(\frac{\mathrm{d}v_3}{\mathrm{d}z}\right)^2 \mathrm{d}z. \tag{52}$$

Let us now show that integrals $I_5 - I_9$ can be interpreted as the corresponding energy integrals of electromagnetic disturbances caused by the surface wave. Introducing

$$\bar{h}_3 = v_3 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{h}_3 e^{-i\xi(x-x^0)}\, \mathrm{d}\xi,$$

$$\bar{h}_1 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{h}_1 e^{-i\xi(x-x^0)}\, \mathrm{d}\xi,$$

$$\bar{e}_2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{e}_2 e^{-i\xi(x-x^0)}\, \mathrm{d}\xi,$$

$$h_1 = \frac{1}{2\pi}\int_{-\infty}^{\infty} h_1 e^{-i\xi(x-x^0)}\, \mathrm{d}\xi,$$

$$h_3 = \frac{1}{2\pi}\int_{-\infty}^{\infty} h_3 e^{-i\xi(x-x^0)}\, \mathrm{d}\xi, \tag{53}$$

one can rewrite the relations (13) and (16) as follows:

$$\hat{h}_1 = -iH\frac{dv_1}{dz}, \quad \hat{h}_3 = \xi H v_1, \quad \hat{e}_2 = \omega\mu H v_1, \tag{54}$$

$$\xi\hat{\hat{e}}_2 = \mu_0\omega\hat{\hat{h}}_3, \quad \frac{d\hat{\hat{h}}_3}{dz} = i\xi\hat{\hat{h}}_1. \tag{55}$$

Using the relations (54) and (55), the integrals for electro-magnetic disturbances (52) can be rewritten as

$$I_7 + \frac{1}{\xi^2}I_9 = \mu_0\int_{-\infty}^{0}(\hat{\hat{h}}_1^2+\hat{\hat{h}}_3^2)\,dz = \mu_0\int_{-\infty}^{0}\hat{\hat{\mathbf{h}}}^2\,dz, \tag{56}$$

$$\xi^2 I_6 + I_8 = \mu\int_{0}^{\infty}(\hat{h}_1^2+\hat{h}_3^2)\,dz = \mu\int_{0}^{\infty}\hat{\mathbf{h}}^2\,dz, \tag{57}$$

$$\frac{\bar{\omega}^2}{\xi^2}I_5 = \varepsilon_0\int_{-\infty}^{0}\hat{\hat{e}}_2^2\,dz = \varepsilon_0\int_{-\infty}^{0}\hat{\hat{\mathbf{e}}}^2\,dz, \tag{58}$$

where vectors $\hat{\mathbf{h}} = [\hat{h}_1, 0, \hat{h}_3]^T$, $\hat{\hat{\mathbf{h}}} = [\hat{\hat{h}}_1, 0, \hat{\hat{h}}_3]^T$ and $\hat{\hat{\mathbf{e}}} = [0, \hat{\hat{e}}_2, 0]^T$ describe the electro-magnetic disturbance in the conducting medium and vacuum respectively. Since the full electro-magnetic energy is essentially all in the magnetic form in a perfect conductor (Landau and Lipshitz, 1960) $(\bar{\omega}^2/\xi^2)/I_5$ represents the full electrical energy and

$$I_7 + \frac{1}{\xi^2}I_9 + \xi^2 I_6 + I_8$$

represents the full magnetic energy of all space under consideration. Relations (56)–(58) reveal that I_5–I_9 are indeed the energy integrals of electromagnetic disturbances caused by the magneto-elastic interactions in a perfectly conducting medium. Overall I_1–I_9 may be called the energy integrals of magneto-elastic Rayleigh waves analogously to the purely elastic case.

We have shown that the magneto-elastic energy integral (50) vanishes at its stationary point for eigenfunctions of magneto-elastic Rayleigh waves similar to the case of the purely elastic problem where the corresponding elastic energy integral (obtained from the explicit Lagrangian) vanishes for eigenfunctions of elastic Rayleigh waves (Aki and Richards, 1980). This similarity delineates the additional advantage of using the single combined operator L [see eqn (19)] which accounts for the whole space (i.e. the conducting half-space and the vacuum), and introducing the scalar product (27) which is rendered self-adjoint with respect to L: one has the opportunity not only to construct the solution to the inhomogeneous equation (18) in the form of eqn (42), but also to get an energy relationship for magneto-elastic Rayleigh waves by rewriting the corresponding homogeneous equation (29) in integral form (50). This integral representation can be very useful for the solution of the homogeneous equation (29) because it allows one to use variational techniques for elastic media (e.g. Rayleigh–Ritz method) to determine the eigenfunctions of magneto-elastic Rayleigh waves.

Moreover, in the problem under consideration, Green's function (41) can be rewritten through the energy integrals (51)–(52). For that purpose we differentiate eqns (44)–(46) with respect to ξ, multiply eqns (44), (45) and (46) by ρv_1^*, ρv_3^*, $(\varepsilon_0\mu_0^2/\xi^2)v_3^*$ respectively, and then integrate the resulting equations with respect to z from $-\infty$ to ∞. Noting that the relation (50) is stationary in accordance with Hamilton's principle for a small variation of eigenfunctions, we can present the result as follows:

$$\|\tilde{\mathbf{V}}\|^2 2\tilde{\omega} \left.\frac{\partial \tilde{\omega}}{\partial \xi}\right|_{\xi=\bar{\xi}} = 2\xi I_2 + I_3 + \xi I_6 + \frac{1}{\xi} I_7. \tag{59}$$

Therefore, $\mathbf{G}_n$, eqn (41), can be rewritten as

$$\mathbf{G}^n = \frac{\tilde{V}^*_{+n}(z^0)\tilde{\mathbf{V}}(z)e^{-i[\xi(x-x^0)-(\pi/2)]}}{2\xi I_2 + I_3 + \xi I_6 + \frac{1}{\xi} I_7}. \tag{60}$$

In Its and Yanovskaya (1979) it has been shown that $2\xi I_2 + I_3$ is proportional to the average flux of elastic energy through a semi-infinite strip of unit width orthogonal to the direction of propagation of Rayleigh waves for one period. A similar statement is valid with respect to an electro-magnetic counterpart $\xi I_6 + (1/\xi) I_7$. Using (54) and (55) one can show that

$$\begin{aligned}
\xi I_6 + \frac{1}{\xi} I_7 &= \xi \mu H^2 \int_0^{\infty} \frac{\hat{h}_3 \hat{e}_2}{\xi H \omega \mu H} \mathrm{d}z + \frac{\mu_0}{\xi} \int_{-\infty}^{0} \frac{\hat{\bar{h}}_3 \xi \hat{\bar{e}}_2}{\mu_0 \omega} \mathrm{d}z \\
&= \frac{1}{\omega} \int_0^{\infty} \hat{h}_3 \hat{e}_2 \, \mathrm{d}z + \frac{1}{\omega} \int_{-\infty}^{0} \hat{\bar{h}}_3 \hat{\bar{e}}_2 \, \mathrm{d}z \\
&= \frac{1}{\omega} \int_0^{\infty} S_1 \, \mathrm{d}z + \frac{1}{\omega} \int_{-\infty}^{0} \bar{S}_1 \, \mathrm{d}z,
\end{aligned} \tag{61}$$

where S_1 and $\bar{S}_1$ are the x-components of the Poynting vectors

$$\mathbf{S} = \mathbf{e} \times \mathbf{h}, \quad \bar{\mathbf{S}} = \bar{\mathbf{e}} \times \bar{\mathbf{h}} \tag{62}$$

in the medium and vacuum respectively. Therefore, one can conclude that $\xi I_6 + (1/\xi) I_7$ describes the flux of electro-magnetic energy through an infinite strip of unit width perpendicular to the direction of propagation of electro-magnetic disturbance caused by the unit line force in the conducting elastic medium for a period.

The denominator in eqn (60) describes the flux of magneto-elastic energy for a cycle of oscillation. Green's function in the form of eqn (60) can be used further for the solution of scattering problems in weakly inhomogeneous conductors as well as for the determination of reflection and transmission coefficients of magneto-elastic Rayleigh waves across strong inhomogeneities such as cracks or inclusions. Some solutions of these problems in purely elastic media (Snieder, 1986; Its and Yanoskaya, 1979) have been obtained based on a simple representation by normalizing eigenfunctions so that the flux elastic energy is equal to unity. In that regard, eqn (60) may provide a more advantageous representation of Green's function for solving problems of scattering and refraction of magneto-elastic surface waves.

5. CONCLUSIONS

Propagation of magneto-elastic Rayleigh waves from a unit line force in a perfectly conducting elastic half-space in a magnetic field is considered in this paper. Green's function of magneto-elastic Rayleigh waves is constructed by using the theory of spectral operators. A unified self-adjoint operator for both the elastic and electro-magnetic disturbances is introduced. The self-adjointedness of the operator is then invoked to construct the solution of the inhomogeneous problem through the eigenfunctions of a corresponding homogeneous problem and to derive energy integrals of magneto-elastic Rayleigh waves similar to the energy integrals known for the elastic waves. Relationships between the elastic and electro-magnetic energy are discussed in some detail. The energy integral representation is presented for the determination of the eigenfunctions of magneto-elastic Rayleigh waves in which methods based on the variational principles can be employed. Because of the simple analytical form, the Green's function constructed in the paper can be used further for solution of different problems of refraction and scattering of magneto-elastic surface waves in inhomogeneous media for possible application in the nondestructive evaluation of advanced electromagnetic materials.

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APPENDIX

Consider a scalar product of two vectors $\mathbf{V} = [V_1, V_2, V_3]^T$ and $\tilde{\mathbf{V}} = [\tilde{V}_1, \tilde{V}_2, \tilde{V}_3]^T$ as

$$S = -\langle\langle \mathbf{L}\mathbf{V} - \omega^2 \mathbf{V}, \tilde{\mathbf{V}} \rangle\rangle, \tag{A1}$$

where the linear operator $\mathbf{L}$ is given by eqn (23). In accordance with the determination of the scalar product (27), S can be rewritten as follows:

$$\begin{aligned} S = {} & k\int_{-\infty}^{0}\left(\frac{\mathrm{d}^2 V_3}{\mathrm{d}z^2} - \xi^2 V_3\right)\tilde{V}_3^*\,\mathrm{d}z + \omega^2 k\int_{-\infty}^{0}\varepsilon_0\mu_0 V_3\tilde{V}_3^*\,\mathrm{d}z \\ & + \int_0^{\infty}\Bigg[-\xi^2(\lambda+2G+\mu H^2)V_1\tilde{V}_1^* - i\xi(\lambda+G)\frac{\mathrm{d}V_2}{\mathrm{d}z}\tilde{V}_1^* \\ & + (G+\mu H^2)\frac{\mathrm{d}^2 V_1}{\mathrm{d}z^2}\tilde{V}_1^* + \rho\omega^2 V_1\tilde{V}_1^* - \xi^2 G V_2\tilde{V}_2^* \\ & + (\lambda+2G)\frac{\mathrm{d}^2 V_2}{\mathrm{d}z^2}\tilde{V}_2^* - i\xi(\lambda+G)\frac{\mathrm{d}V_1}{\mathrm{d}z}\tilde{V}_2^* + \rho\omega^2 V_2\tilde{V}_2^*\Bigg]\mathrm{d}z. \end{aligned} \tag{A2}$$

Using integration by parts the above integral leads to $\mathbf{L}$ from vector $\mathbf{V}$ to $\tilde{\mathbf{V}}$:

$$\begin{aligned} S = {} & k\left(\frac{\mathrm{d}V_3}{\mathrm{d}z}\tilde{V}_3^* - V_3\frac{\mathrm{d}\tilde{V}_3^*}{\mathrm{d}z}\right)\Bigg|_{-\infty}^{0} + k\int_{-\infty}^{0}\left(V_3\frac{\mathrm{d}^2\tilde{V}_3^*}{\mathrm{d}z^2} - \xi^2 V_3\tilde{V}_3^*\right)\mathrm{d}z \\ & + \omega^2 k\mu_0\int_{-\infty}^{0} V_3\tilde{V}_3^*\,\mathrm{d}z + \int_0^{\infty}\Bigg[-\xi^2(\lambda+2G+\mu H^2)V_1\tilde{V}_1^* \\ & + i\xi(\lambda+G)V_2\frac{\mathrm{d}\tilde{V}_1^*}{\mathrm{d}z} + (G+\mu H^2)V_1\frac{\mathrm{d}^2\tilde{V}_1^*}{\mathrm{d}z^2} + \rho\omega^2 V_1\tilde{V}_1^* \\ & - \xi^2 G V_2\tilde{V}_2^* + (\lambda+2G)V_2\frac{\mathrm{d}^2\tilde{V}_2^*}{\mathrm{d}z^2} + i\xi(\lambda+G)V_1\frac{\mathrm{d}\tilde{V}_2^*}{\mathrm{d}z} + \rho\omega^2 V_2\tilde{V}_2^*\Bigg]\mathrm{d}z \\ & + \Bigg[-i\xi(\lambda+G)V_2\tilde{V}_1^* + (G+\mu H^2)\left(\frac{\mathrm{d}V_1}{\mathrm{d}z}\tilde{V}_1^* - V_1\frac{\mathrm{d}\tilde{V}_1^*}{\mathrm{d}z}\right) \\ & - i\xi(\lambda+G)V_1\tilde{V}_2^* + (\lambda+2G)\left(\frac{\mathrm{d}V_2}{\mathrm{d}z}\tilde{V}_2^* - V_2\frac{\mathrm{d}\tilde{V}_2^*}{\mathrm{d}z}\right)\Bigg]\Bigg|_0^{\infty} \\ = {} & -\langle\langle \mathbf{V}, \mathbf{L}\tilde{\mathbf{V}} - \omega^2\tilde{\mathbf{V}}\rangle\rangle + B_1 + B_2, \end{aligned}$$

where nonintegral terms can be rewritten taking into account the decrease in $\mathbf{V}$ for $z \to \pm\infty$:

$$B_1 = k\left(\frac{\mathrm{d}V_3}{\mathrm{d}z}\tilde{V}_3^* - V_3\frac{\mathrm{d}\tilde{V}_3^*}{\mathrm{d}z}\right)(0), \tag{A3}$$

$$\begin{aligned} B_2 = -\Bigg[& -i\xi(\lambda+G)V_2\tilde{V}_1^* + (G+\mu H^2)\left(\frac{\mathrm{d}V_1}{\mathrm{d}z}\tilde{V}_1^* - V_1\frac{\mathrm{d}\tilde{V}_1^*}{\mathrm{d}z}\right) \\ & - i\xi(\lambda+G)V_1\tilde{V}_2^* + (\lambda+2G)\left(\frac{\mathrm{d}V_2}{\mathrm{d}z}\tilde{V}_2^* - V_2\frac{\mathrm{d}\tilde{V}_2^*}{\mathrm{d}z}\right)\Bigg](0). \end{aligned} \tag{A4}$$

Inserting (24) into (A4) leads to the relation:

$$B_2 = -\left[\left((\lambda+2G)\frac{\mathrm{d}V_2}{\mathrm{d}z} - i\xi\lambda V_1 + \mu H^2 i\xi V_1\right)\tilde{V}_2^* - \left((\lambda+2G)\frac{\mathrm{d}\tilde{V}_2^*}{\mathrm{d}z} + i\xi\lambda\tilde{V}_1^* - \mu H^2 i\xi\tilde{V}_1^*\right)V_2\right](0).$$

Taking into account (25) one can rewrite the last relation as

$$B_2 = [-\mu H V_3\tilde{V}_2^* + \mu H\tilde{V}_3^* V_2](0) \tag{A5}$$

or

$$B_2 = \frac{\mu H}{i\xi}\left(V_3\frac{\mathrm{d}\hat{V}_1^*}{\mathrm{d}z} + \hat{V}_3^*\frac{\mathrm{d}V_1}{\mathrm{d}z}\right)(0). \tag{A6}$$

Finally, we take $B = B_1 + B_2$ by the use of eqn (26) as follows:

$$\begin{aligned} B &= \hat{V}_3^*(0)\left(k\frac{\mathrm{d}V_3}{\mathrm{d}z} + \frac{\mu H}{i\xi}\frac{\mathrm{d}V_1}{\mathrm{d}z}\right) - V_3(0)\left(k\frac{\mathrm{d}\hat{V}_3^*}{\mathrm{d}z} - \frac{\mu H}{i\xi}\frac{\mathrm{d}\hat{V}_1^*}{\mathrm{d}z}\right) \\ &= \hat{V}_3^*\frac{\mathrm{d}V_1}{\mathrm{d}z}\left(i\xi Hk\frac{\mu}{\mu_0} + \frac{\mu H}{i\xi}\right) - V_3\frac{\mathrm{d}\hat{V}_1^*}{\mathrm{d}z}\left(-i\xi Hk\frac{\mu}{\mu_0} - \frac{\mu H}{i\xi}\right) \\ &= \frac{\mu H i\xi}{\mu_0}\left(k - \frac{\mu_0}{\xi^2}\right)\left(\hat{V}_3^*\frac{\mathrm{d}V_1}{\mathrm{d}z} + V_3\frac{\mathrm{d}\hat{V}_1^*}{\mathrm{d}z}\right). \end{aligned} \tag{A7}$$

From the last expression in the above equation, B becomes identically zero when $k = \mu_0/\xi^2$ or

$$\langle\langle \mathbf{L}\mathbf{V} - \omega^2\mathbf{V}, \hat{\mathbf{V}}\rangle\rangle = \langle\langle \mathbf{V}, \mathbf{L}\hat{\mathbf{V}} - \omega^2\hat{\mathbf{V}}\rangle\rangle.$$

Therefore, $\mathbf{L}$ is a self-adjoint operator with respect to the scalar product (27) provided that $k = \mu_0/\xi^2$.